

Properties of the gauge invariant quark Green's function in two-dimensional QCD

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Abstract

Using an exact integrodifferential equation we study the properties of the gauge invariant quark Green's function, defined with a path-ordered gluon field phase factor along a straight line, in two-dimensional QCD in the large- N_c limit. The Green's function is found to be infrared finite with singularities represented by an infinite number of threshold type branch points with a power equal to $-3/2$, starting at positive mass squared values. The solution is analytically determined.

Keywords: QCD, quark, gluon, Wilson loop, gauge invariant Green's function.

Gauge invariant quark Green's functions are defined with the aid of path-ordered gluon field phase factors [1, 2]. Skew-polygonal lines for the paths are of particular interest since they can be represented as junctions of simpler straight line segments. For such lines with n sides and $n - 1$ junction points y_1, y_2, \dots, y_{n-1} between the segments, we define the Green's function $S_{(n)}$ as

$$S_{(n)}(x, x'; y_{n-1}, \dots, y_1) = -\frac{1}{N_c} \langle \bar{\psi}(x') U(x', y_{n-1}) \times U(y_{n-1}, y_{n-2}) \dots U(y_1, x) \psi(x) \rangle, \quad (1)$$

where $U(x, y)$ is a path-ordered phase factor along a straight line segment joining y to x . The simplest such Green's function corresponds to $n = 1$, for which the points x and x' are joined by a single straight line:

$$S_{(1)}(x, x') \equiv S(x, x') = -\frac{1}{N_c} \langle \bar{\psi}(x') U(x', x) \psi(x) \rangle. \quad (2)$$

(We shall generally omit the index 1 from that function.)

The theory is quantized in two steps. First, one integrates with respect to the quark fields. This produces in various terms the quark propagator in the presence of the gluon field. Then one integrates with respect to the gluon field through Wilson loops [3–8]. To achieve

the latter operation, we use for the quark propagator in external field a representation which involves phase factors along straight lines together with the full gauge invariant quark Green's function [9, 10]. This representation is a generalization of the one introduced by Eichten and Feinberg when calculating the relativistic effects starting from a nonrelativistic limit [11].

The quark propagator in the external gluon field is expanded around the following gauge covariant quantity:

$$\left[\tilde{S}(x, x') \right]_b^a \equiv S(x, x') \left[U(x, x') \right]_b^a. \quad (3)$$

It is possible to set up an integral equation realizing the previous expansion. Its systematic use leads to the derivation of functional relations between the Green's functions $S_{(n)}$ (skew-polygonal line with n segments) and S (one segment).

Using then the equations of motion relative to the Green's functions, one establishes the following equation for $S(x, x')$ [9]:

$$\begin{aligned} (i\gamma \cdot \partial_{(x)} - m)S(x, x') &= i\delta^4(x - x') \\ &+ i\gamma^\mu \left\{ K_{2\mu}(x', x, y_1) S_{(2)}(y_1, x'; x) \right. \\ &+ \sum_{n=3}^{\infty} K_{n\mu}(x', x, y_1, \dots, y_{n-1}) \\ &\quad \left. \times S_{(n)}(y_{n-1}, x'; x, y_1, \dots, y_{n-2}) \right\}, \quad (4) \end{aligned}$$

where the kernel K_n ($n = 2, 3, \dots$) contains globally n derivatives of Wilson loop averages with skew-

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polygonal contours and also the Green's function S and its derivative. (Integrations on intermediate variables are implicit.) The Green's functions $S_{(n)}$ being themselves related to the simplest Green's function S through series expansions resulting from functional relations, (4) is ultimately an integrodifferential equation for S . One expects that the kernels with small numbers of derivatives will provide the leading contributions. Therefore, the first kernel K_2 in (4) would contain the driving term of the interaction.

Equation (4) shows that the Green's function S should have singularities in momentum space, generated by the free quark propagator (the inverse of the Dirac operator in the left-hand side). On the other hand, such singularities cannot be generated by saturating the Green's function with intermediate states made of hadrons, which are color singlets; the intermediate states that saturate the Green's function are necessarily colored states. It is therefore necessary to assume that quarks and gluons, which are colored objects, continue forming a complete set of states for saturation schemes and contribute, as the building blocks of the theory, with positive energies. It is only the solution of eq. (4) which should provide the indication about the issue of their physical status in observable phenomena. The above hypothesis has an immediate consequence about the analytic properties of the Green's function: it satisfies a generalized form of the Källén–Lehmann representation [9, 12–16].

As a first attempt to solve eq. (4), we have considered the case of two-dimensional QCD in the large- N_c limit [17–19]. That theory is expected to have similar properties as four-dimensional QCD concerning its confining aspect; furthermore, asymptotic freedom is realized there rather trivially, since the theory is superrenormalizable. In two dimensions, the logarithm of the Wilson-loop average of a simple contour is proportional to the area enclosed by the contour [20–22]. Equation (4) considerably simplifies and it is only the lowest-order kernel K_2 that survives. Additional simplification arises from the fact that it involves a functional second-order derivative of the Wilson-loop average, which in two dimensions reduces to a delta-function. Taking also into account translation invariance, eq. (4) becomes [23]:

$$\begin{aligned} (i\gamma \cdot \partial - m)S(x) &= i\delta^2(x) \\ &- \sigma \gamma^\mu (g_{\mu\alpha} g_{\nu\beta} - g_{\mu\beta} g_{\nu\alpha}) x^\nu x^\beta \\ &\times \left[\int_0^1 d\lambda \lambda^2 S((1-\lambda)x) \gamma^\alpha S(\lambda x) \right. \\ &\left. + \int_1^\infty d\xi S((1-\xi)x) \gamma^\alpha S(\xi x) \right], \end{aligned} \quad (5)$$

where σ is the string tension.

This equation is solved by decomposing S into Lorentz invariant parts:

$$S(p) = \gamma \cdot p F_1(p^2) + F_0(p^2), \quad (6)$$

or, in x -space:

$$S(x) = \frac{1}{2\pi} \left(\frac{i\gamma \cdot x}{r} \widetilde{F}_1(r) + \widetilde{F}_0(r) \right), \quad r = \sqrt{-x^2}. \quad (7)$$

One obtains, with the introduction of the Lorentz invariant functions, two coupled equations. Their resolution proceeds through several steps, mainly based on the analyticity properties resulting from the spectral representation of the Green's function [23]. The solutions are obtained in explicit form for any value of the quark mass m .

The covariant functions $F_1(p^2)$ and $F_0(p^2)$ are, for complex p^2 :

$$F_1(p^2) = -i \frac{\pi}{2\sigma} \sum_{n=1}^{\infty} b_n \frac{1}{(M_n^2 - p^2)^{3/2}}, \quad (8)$$

$$F_0(p^2) = i \frac{\pi}{2\sigma} \sum_{n=1}^{\infty} (-1)^n b_n \frac{M_n}{(M_n^2 - p^2)^{3/2}}. \quad (9)$$

The masses M_n ($n = 1, 2, \dots$) have positive values greater than the quark mass m and are labelled with increasing values with respect to n ; their squares represent the locations of branch point singularities with power $-3/2$. The masses M_n and the coefficients b_n satisfy an infinite set of coupled algebraic equations that are solved numerically. Their asymptotic behaviors for large n , such that $\sigma\pi n \gg m^2$, are:

$$M_n^2 \simeq \sigma\pi n, \quad b_n \simeq \frac{\sigma^2}{M_n}. \quad (10)$$

In x -space, the solutions are:

$$\widetilde{F}_1(r) = \frac{\pi}{2\sigma} \sum_{n=1}^{\infty} b_n e^{-M_n r}, \quad (11)$$

$$\widetilde{F}_0(r) = \frac{\pi}{2\sigma} \sum_{n=1}^{\infty} (-1)^{n+1} b_n e^{-M_n r}. \quad (12)$$

[$r = \sqrt{-x^2}$.]

We present in Fig. 1 the function iF_0 for spacelike p and in Fig. 2 its real part for timelike p , for the case $m = 0$.

In conclusion, the spectral functions of the quark Green's function are infrared finite and lie on the positive real axis of p^2 . No singularities in the complex plane or on the negative real axis have been found. This

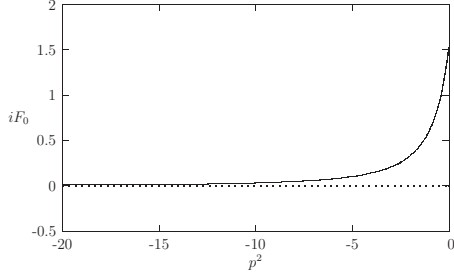


Figure 1: The function iF_0 for spacelike p , in mass unit of $\sqrt{\sigma}/\pi$, for $m = 0$.

means that quarks contribute like physical particles with positive energies. (In two dimensions there are no physical gluons.)

The singularities of the Green's function are represented by an infinite number of threshold type singularities, characterized by a power of $-3/2$ and positive masses M_n ($n = 1, 2, \dots$). The corresponding singularities are stronger than simple poles and this feature is an indication about the difficulty in the observability of quarks as asymptotic states.

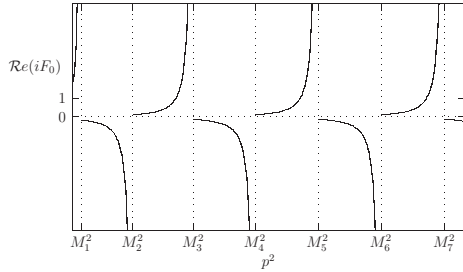


Figure 2: The real part of the function iF_0 for timelike p , in mass unit of $\sqrt{\sigma}/\pi$, for $m = 0$.

The threshold masses M_n represent dynamically generated masses and maintain the scalar part of the Green's function at a nonzero value even when the quark mass is zero.

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